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WEAK LINKING AND MULTIPLICITIES

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Introduction

In this paper we study the notion of weak linking (see Definition 1.7 and its connection to Serre's intersection multiplicity. Theorem 1.9 provides us with a necessary and sufficient condition for weak linking between two Cohen–Macaulay modules of codimension 1 over a Gorenstein ring in terms of syzygies of the respective modules. An immediate corollary (see 1.12) of this theorem is the fact that modules of finite length, finite projective dimension over a Gorenstein ring R of dimension 1 are weakly linked to $R/(x)$, x a non-zero-divisor on R. Some more corollaries are described in 1.11, 1.13 and 1.14 – the latter is of particular interest, since it describes the modules of finite length over $K[[x^2, x^{2n+1}]]$ via weak linking.

In Proposition 2.1 we give a necessary and sufficient condition for two C–M modules over a C–M ring to be isomorphic in terms of their syzgies. This eventually leads in 2.2 to the higher dimensional analogue of Theorem 1.9 which I cannot prove yet. But in 2.2 we show how this higher dimensional analogue implies Serre's intersection multiplicity conjecture (in a more general set-up) when the sum of the dimensions of the modules is less than that of the ring. In 2.10 we answer the following question partially: Given a complete intersection R with $\dim R = 1$, M a module with $l(M) < \infty$, N a module with $S^{-1}N$ $S^{-1}R$-free, where $S = R - \bigcup P_i$, $P_i \in \mathrm{Ass}(R)$, is it true that $l(M \otimes_R N) \geq r(N) l(M)$ where $r(N)$ is the $S^{-1}R$ rank of $S^{-1}N$. We show that an affirmative answer to the above question at least when $\mathrm{Tor}_i^R(M, N) = 0$, for all $i > 0$, would imply the intersection multiplicity conjecture for a pair of modules with sum of their dimensions less than that of the ring.

In this work all rings are commutative local with identity and all modules are finitely generated.

We say that a local ring R satisfies the vanishing conjecture if given any pair of modules M and N such that

$$l(M \otimes_R N) < \infty, \qquad \mathrm{pd}_R M < \infty, \qquad \dim M + \dim N < \dim R,$$

$$\chi(M, N) = \sum_{i=0}^{\mathrm{pd}\, M} (-1)^i l(\mathrm{Tor}_i^R(M, N)) = 0.$$

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The reader can notice immediately that this is a generalized version of Serre's intersection multiplicity over a regular local ring.

1. Weak linking

1.1. Notation. We use the following abbreviations and notations:

$l(M)$ = length of the module M,
$\mathbb{Q}$ = the field of rational numbers,
C–M = Cohen–Macaulay,
n.z.d. = non-zero divisor,
d.v.r. = discrete valuation ring,
$Q\{R\}$ = total quotient ring of the ring R,
pd M = projective dimension of M,
$r(N)$ = torsion-free rank of N,
$Q\{N\} = N \otimes_R \{Q\}R$,
$P^{(n)} = \{x \in R \mid tx \in p^n \text{ for some } t \in R - P\}$.

Let R be a local ring with 1. Let M be a finitely generated module with finite projective dimension n. Let N be another finitely generated module with $l(M \otimes N) < \infty$. Then

$$\chi_i(M, N) = \sum_{k=0}^{n-i} (-1)^k l(\mathrm{Tor}_{i+k}^R(M, N)).$$

We leave the proof of the following lemma as an exercise for the reader.

1.2. Special Lemma. *Let R be a local ring with* 1. *Let M be a finitely generated module with* pd $M = n$. *Let N be another with $M \otimes_R N \neq 0$ and let* $\mathrm{Ann}_R M$ *contain an N-sequence of length r. Then* $\mathrm{Tor}_{n-i}^R(M, N) = 0$, $0 \le i < r$.

1.3. Lemma. *Let R be a local ring. We consider an exact sequence*

$$0 \to S \xrightarrow{f} R^n \to M \to 0.$$

Let $\phi : R^n \to R^n$ be given by

$$\phi(e_1) = xe_1, \quad \phi(e_2) = e_2, \quad \ldots, \quad \phi(e_n) = e_n,$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 *on the i-th place and x is an n.z.d. in R. Then ϕ is injective and there exist exact sequences*

$$0 \to S \xrightarrow{\phi f} R^n \to M^1 \to 0, \qquad 0 \to M \to M^1 \to \frac{R}{(x)} \to 0.$$

Proof. Obvious.

1.4. Lemma. *Consider an exact sequence*

$$0 \to S \xrightarrow{f} R^n \to M \to 0.$$

Let $\phi : R^n \to R^n$ *be given by*

$$\phi(e_1) = e_1, \quad \phi(e_2) = e_2 - x_{12}e_1, \quad \phi(e_3) = e_3 - x_{13}e_1, \ \ldots, \ \phi(e_n) = e_n - x_{1n}e_1.$$

Then ϕ *is an isomorphism. Let* $M^1 = \operatorname{coker}(\phi \cdot f)$. *Then* $M \simeq M^1$.

Proof. Obvious.

1.5. Lemma. *Let R be a local ring. Let* $A = (a_{ij})$ *e an* $n \times n$ *matrix over R such that* $\det A$ *is an n.z.d. Then after column operations* a_{11} *can be replaced by an n.z.d.*

Proof. See [4].

1.6. Lemma. *Consider an exact sequence*

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Let $\operatorname{syz}^i M_1, \operatorname{syz}^i M_3$ *be any specific choices of i-th syzygies for* M_1, M_3 *(not necessarily minimal). Then*

$$0 \to \operatorname{syz}^i M_1 \to \operatorname{syz}^i M_2 \to \operatorname{syz}^i M_3 \to 0$$

is exact for some choice of $\operatorname{syz}^i M_2$.

Proof. Obvious.

It is clear that if $\operatorname{pd} M_1 = i_0$, by taking a minimal resolution of M_1,

$$\operatorname{syz}^{i_0+1} M_2 \simeq \operatorname{syz}^{i_0+1} M_3,$$

while if $\operatorname{pd} M_3 = i_0$,

$$\operatorname{syz}^{i_0+1} M_1 \oplus \operatorname{syz}^{i_0+1} M_3 \simeq \operatorname{syz}^{i_0} M_2.$$

Finally if $\operatorname{pd} M_2 = i_0$,

$$\operatorname{syz}^{i_0} M_1 = \operatorname{syz}^1(\operatorname{syz}^{i_0} M_3) = \operatorname{syz}^{i_0+1} M_3$$

(since $\operatorname{syz}^{i_0} M_2$ is free).

1.7. Definitions. Let R be a C–M ring. We say that two modules M_1 and M_2 are t-linked on the right, or that M_2 and M_1 are t-linked on the right, if there exists an exact sequence $0 \to M_1 \to M_2 \to E \to 0$ where E is a finite direct sum of cyclic modules of the form

$$\frac{R}{(x_1, \ldots, x_t)R},$$

where $\{x_1, \ldots, x_t\}$ is an R-sequence. Let $\wedge$ be the equivalence relation generated by 'being t-linked on the right' in the category of finitely generated modules over R. We say two modules M_1, M_2 are *weakly t-linked on the right* if they belong to the same class under $\wedge$; similarly we define *weak t-linkage on the left* (respectively *in the middle*) by placing E on the left (respectively in the middle) in the above sequence.

We say M_1 and M_2 are *weakly t-linked at the end* if they belong to the same class defined by the equivalence relation generated by 'being t-linked on the right' and 'being t-linked on the left'.

M_1 and M_2 are *weakly t-linked* if they belong to the same class defined by the equivalence relation generated by 'being t-linked on the right', 'being t-linked on the left' and 'being t-linked in the middle'.

1.8. Notations. (i) We write $M_1 \sim_t M_2$ to express that M_1 and M_2 are weakly t-linked. We write $M_1 \sim_t M_2$ at the end to express that M_1 and M_2 are weakly t-linked at the end.

(ii) For a module T, we denote by $\langle T \rangle$ the projective class of T, i.e. all modules N such that $N \oplus R^n = T \oplus R^t$ for some $n \geq 0$, $t \geq 0$.

1.9. Theorem. *Let R be a Gorenstein ring of dimension n. Let M_1, M_2 be two C–M modules with* $\dim M_1 = \dim M_2 = n-1$. *Then*

(i) $M_1 \sim_1 M_2$ *at the end* $\Leftrightarrow$ $\langle \mathrm{syz}^i(M_1) \rangle = \langle \mathrm{syz}^i(M_2) \rangle$ *for some* $i > 0$.

(ii) $M_1 \sim_1 M_2$ $\Leftrightarrow$ $\langle \mathrm{syz}^i(M_1) \rangle = \langle \mathrm{syz}^{i+k}(M_2) \rangle$ *for some* $k > 0$, $i > 0$.

Proof. (i)($\Rightarrow$) From an exact sequence of the form

$$0 \to M \to N \to \bigoplus_{i=1}^{t} \frac{R}{(x_i)} \to 0,$$

we get by Lemma 1.6

$$\mathrm{syz}^1(N) \cong \mathrm{syz}^1(M) \oplus R^t,$$

i.e. $\langle \mathrm{syz}^1(N) \rangle = \langle \mathrm{syz}^1(M) \rangle$. From an exact sequence of the form

$$0 \to \bigoplus_{i=1}^{t} \frac{R}{x_i R} \to M \to N \to 0,$$

we get $\mathrm{syz}^2(M) \cong \mathrm{syz}^2(N)$ by Lemma 1.6, i.e. $\langle \mathrm{syz}^2(M) \rangle \cong \langle \mathrm{syz}^2(N) \rangle$. We note that if $\langle \mathrm{syz}^i(M_1) \rangle = \langle \mathrm{syz}^i(M_2) \rangle$ then $\langle \mathrm{syz}^{i+j}(M_1) \rangle = \langle \mathrm{syz}^{i+j}(M_2) \rangle$, $j \geq 0$. Hence $M_1 \sim_1 M_2$ at the end implies $\langle \mathrm{syz}^i(M_1) \rangle = \langle \mathrm{syz}^i(M_2) \rangle$ for some $i > 0$.

(i)($\Leftarrow$) First let us assume $\langle \mathrm{syz}^1(M_1) \rangle = \langle \mathrm{syz}^1(M_2) \rangle$. We consider

$$0 \to S_1 \xrightarrow{f_1} R^{n_1} \to M_1 \to 0, \tag{1'}$$

$$0 \to S_2 \xrightarrow{f_2} R^{n_2} \to M_2 \to 0. \tag{2'}$$

Since (a) $S_1 \oplus R^{t_1} \cong S_2 \oplus R^{t_2}$; and (b) $T^{-1}S_i$ is a free $T^{-1}R$ module of rank $n_i, i=1,2$, where $T = R - \bigcup P_i$, $P_i \in \mathrm{Ass}(R)$; we have $n_1 + t_1 = n_2 + t_2$. From (1′)

$$0 \to S_1 \oplus R^{t_1} \xrightarrow{f_1 \oplus \mathrm{id}} R^{n_1} \oplus R^{t_1} \to M_1 \to 0, \tag{1″}$$

$$0 \to S_2 \oplus R^{t_2} \xrightarrow{f_2 \oplus \mathrm{id}} R^{n_2} \oplus R^{t_2} \to M_2 \to 0. \tag{2″}$$

Hence (from (1″) and (2″)) without loss of generality we can write

$$0 \to S \xrightarrow{f} R^n \to M_1 \to 0, \tag{1}$$

$$0 \to S \xrightarrow{g} R^n \to M_2 \to 0. \tag{2}$$

Let $f_1, \ldots, f_n$ be the components of f; let $g_1, \ldots, g_n$ be the components of g. Since

$$T^{-1}S \xrightarrow[\sim]{T^{-1}(f)} T^{-1}(R^n),$$

$T^{-1}(f)^*$: $\mathrm{Hom}(T^{-1}R^n, T^{-1}R) \xrightarrow{\cong} \mathrm{Hom}(T^{-1}S, T^{-1}R)$. Thus $\mathrm{Hom}(T^{-1}S, T^{-1}R)$ is a free $T^{-1}R$ module of rank n generated by $f_1, \ldots, f_n$.

Similarly $\mathrm{Hom}(T^{-1}S, T^{-1}R)$ is a free $T^{-1}R$ module of rank n generated by $g_1, \ldots, g_n$.

Since $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ are two sets of bases for $\mathrm{Hom}(T^{-1}S, T^{-1}R)$ as a $T^{-1}R$ module we have

$$\begin{pmatrix} g_n \\ \vdots \\ g_n \end{pmatrix} = \tilde{A} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

where $\tilde{A}$ is an $n \times n$ matrix over $T^{-1}R$, and $\det \tilde{A}$ is a unit in $T^{-1}R$. Choosing the denominators of the entries of $\tilde{A}$ we have

$$r \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} = A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

where entries of A are in R, r and $\det A$ are n.z.d.s. in R. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Then

$$rg_i = \sum_{j=1}^{n} a_{ij} f_j. \tag{3}$$

For every i we denote the i-th row of A by α_i. We write

$$\sum_{j=1}^{n} a_{ij} f_j = \alpha_i \cdot f, \quad f = (f_1, \ldots, f_n).$$

The proof will be completed by the following steps:

Step 1. From (2), since $g(s)=(g_1(s),\dots,g_n(s))\in R^n, s\in S$, we have

$$M_2 \cong \frac{R^n}{(g_1,\dots,g_n)S}. \tag{4}$$

Consider the map $\phi: R^n \to R^n$ given by

$$e_i \mapsto re_i \quad \forall i.$$

Then by repeated application of Lemma 1.3,

$$\frac{R^n}{(g_1,\dots,g_n)S} \sim_1 \frac{R^n}{(rg_1,\dots,rg_n)S} \quad \text{at the end}$$

$$= \frac{R^n}{(\alpha_1\cdot f,\dots,\alpha_n\cdot f)S} \quad \text{(by (3))}. \tag{5}$$

In the matrix A, by Lemmas 1.4 and 1.5, we can assume that a_{11} is an n.z.d. Consider the map $\phi_1: R^n \to R^n$ given by

$$\phi_1(e_1)=e_1, \qquad \phi_1(e_2)=a_{11}e_2,\ \dots,\ \phi_1(e_n)\to a_{11}e_n,$$

then by Lemma 1.3,

$$\frac{R^n}{(\alpha_1\cdot f,\dots,\alpha_n\cdot f)S} \sim_1 \frac{R^n}{(\alpha_1\cdot f, a_{11}(\alpha_2\cdot f),\dots,a_{11}(\alpha_n\cdot f))S}. \tag{6}$$

We consider the map $\phi_2: R^n \to R^n$ defined by

$$\phi_2(e_1)=e_1-a_{21}e_2-\cdots-a_{n1}e_n, \qquad \phi_2(e_2)=e_2,\ \dots,\ \phi_2(e_n)=e_n.$$

Then by Lemma 1.4,

$$\frac{R^n}{(\alpha_1\cdot f, a_{11}(\alpha_2\cdot f),\dots,a_{11}(\alpha_n\cdot f))S} \xrightarrow{\ \sim\ } \frac{R^n}{(\alpha_1\cdot f, \beta_2\cdot f', \beta_3\cdot f,\dots,\beta_n\cdot f')S} \tag{7}$$

where $f'=(f_2,\dots,f_n)$, $\beta_i=a_{11}\alpha_1-a_{i1}\alpha_1$, i.e., $\beta_2,\dots,\beta_n$ are the rows of the matrix

$$B=\begin{pmatrix} a_{11}a_{22}-a_{12}a_{21} & \cdots & a_{11}a_{2n}-a_{21}a_{12} \\ \vdots & & \\ a_{11}a_{n2}-a_{n1}a_{n2} & \cdots & a_{11}a_{nn}-a_{n1}a_{1n} \end{pmatrix}.$$

Since $\det A$ is an n.zd. $\det B$ is also. We write $B=(b_{ij})$, $1\le i\le n-1$, $1\le j\le n-1$. Since $\det B$ is an n.z.d. in R by Lemma 1.5, we can assume b_{11} is an n.z.d. in R. We denote the module on the right hand side of (7) by N. Now starting with B and repeating the same process a finite number of times we see that

$$N \sim_1 \frac{R^n}{(\lambda_{ij})(f)S} \quad \text{at the end} \tag{8}$$

where (λ_{ij}) is an upper triangular matrix and $\det(\lambda_{ij})=\lambda_{11}\cdot\lambda_{22}\cdots\lambda_{nn}$ is an n.z.d. in R.

We denote the module on the right-hand side of (8) by T. By (4)–(8), we have shown $M_2\sim_1 T$ at the end.

Step 2. We have

$$M_1\cong\frac{R^n}{(f_1,\dots,f_n)S}. \tag{9}$$

We consider the map $\psi_1: R^n\to R^n$ defined by

$$\psi_1(e_1)=\lambda_{11}e_1,\quad \psi_1(e_2)=e_2,\ \dots,\ \psi_1(e_n)=e_n.$$

Then by Lemma 1.3,

$$\frac{R^n}{(f_1,\dots,f_n)S}\sim_1\frac{R^n}{(\lambda_{11}f_1,f_2,\dots,f_n)S}.$$

We apply the map $\psi_2: R^n\to R^n$ given by

$$\begin{aligned}&e_1\mapsto e_1,\\ &e_2\mapsto e_2+\lambda_{12}e_1,\\ &\vdots\\ &e_n\mapsto e_n+\lambda_{1n}e_1.\end{aligned}$$

Then by Lemma 1.4,

$$\frac{R^n}{(\lambda_{11}f_1,f_2,\dots,f_n)S}\xrightarrow{\sim}\frac{R^n}{(\lambda_{11}f_1+\cdots+\lambda_{1n}f_n,f_2,\dots,f_n)S}\quad\text{at the end.} \tag{10}$$

Now repeating the above process a finite number of times we see

$$M_1\sim_1\frac{R^n}{(\lambda_{11}f_1+\cdots+\lambda_{1n}f_n,\lambda_{22}f_2+\cdots+\lambda_{2n}f_n,\dots,\lambda_{nn}f_n)S}\quad\text{at the end}$$

Hence from Step 1 and Step 2, $M_1\sim_1 M_2$ at the end.

Now suppose $\langle\mathrm{syz}^i(M_1)\rangle=\langle\mathrm{syz}^i(M_2)\rangle$ for $i>1$. By applying arguments similar to those applied at the beginning of the proof, we can write

$$0\to S\to R^{n_{i-1}}\to\cdots\to R^{n_1}\xrightarrow{\alpha}R^{n_0}\to M_1\to 0, \tag{11}$$

$$0\to S\to R^{m_{i-1}}\to\cdots\to R^{m_1}\xrightarrow{\beta}R^{m_0}\to M_2\to 0. \tag{12}$$

Let $S_1=\operatorname{Im}\alpha$, $S_2=\operatorname{Im}\beta$. Then we have

$$0\to S\to R^{n_{i-1}}\to\cdots\to R^{n_1}\to S_1\to 0, \tag{13}$$

$$0\to S\to R^{m_{i-1}}\to\cdots\to R^{m_1}\to S_2\to 0. \tag{14}$$

Applying Hom($,R) = {}^{\upsilon}$ we get

$$0 \to S_1^{\upsilon} \to R^{n_1^{\upsilon}} \to \cdots \to R^{n_{i-1}^{\upsilon}} \to S^{\upsilon} \to 0, \tag{13'}$$

$$0 \to S_2^{\upsilon} \to R^{m_1^{\upsilon}} \to \cdots \to R^{m_{i-1}^{\upsilon}} \to S^{\upsilon} \to 0. \tag{14'}$$

Since $(R^t)^{\upsilon} \cong R^t$, from (13′) and (14′) we get the following homomorphism:

$$S_1^{\upsilon} \oplus R^p \cong S_2^{\upsilon} \oplus R^q. \tag{15}$$

Since M_1, M_2 are C–M modules with $\dim M_1 = \dim M_2 = n-1$, S_1 and S_2 are C–M modules and $\dim S_1 = \dim S_2 = n$, S_1 and S_2 are reflexive, i.e. $S_1 \cong S_1^{\upsilon\upsilon}$, $S_2 \cong S_2^{\upsilon\upsilon}$. Applying Hom($,R$) in (15), $S_1^{\upsilon\upsilon} \oplus R^{p^{\upsilon}} \cong S_2^{\upsilon\upsilon} \oplus R^{q^{\upsilon}}$. Thus $S_1 \oplus R^p \cong S_2 \oplus R^{\upsilon}$, i.e. $\langle S_1 \rangle = \langle S_2 \rangle$, i.e. $\langle \mathrm{syz}^1(M_1) \rangle = \langle \mathrm{syz}^1(M_2) \rangle$ and hence by the first part $M_1 \sim_1 M_2$ at the end.

Remark. We note that throughout the proof the linking was always actually on the right.

(ii)(⇒) Suppose $M_1 \sim_1 M_2$. We consider the following exact sequence:

$$0 \to M \to \sum_{i=1}^{t} \frac{R}{x_i R} \to N \to 0, \quad x \text{ an n.z.d. in } R.$$

Then by Lemma 1.6, we have $\langle \mathrm{syz}^1(M) \rangle \cong \langle \mathrm{syz}^2(N) \rangle$. When $M_1 \sim_1 M_2$, since only a finite number of such exact sequences and a finite number of exact sequences as described in (i) occur, we have from the (⇒) part in (i) and from the above, $\langle \mathrm{syz}^i(M_1) \rangle = \langle \mathrm{syz}^{i+k}(M_2) \rangle$ for some $k > 0$, $i > 0$.

(ii)(⇐) Suppose $\langle \mathrm{syz}^i(M_1) \rangle = \langle \mathrm{syz}^{i+k}(M_2) \rangle$. Since $\dim M_2 = n-1$, and R is C–M we have depth $\mathrm{Ann}_R M_2 = 1$. Let $x \in \mathrm{Ann}_R M_2$ be an n.z.d. We map direct sums of R/xR onto M_2. If we do this once we get $0 \to M_2^{(1)} \to E \to M_2 \to 0$ and $\langle \mathrm{syz}^j M_2 \rangle = \langle \mathrm{syz}^{j-1} M_2^{(1)} \rangle$, for large j. After k steps we get $M_2^{(k)} \sim_1 M_2$ and

$$\begin{aligned} \langle \mathrm{syz}^{r+k}(M_2) \rangle &= \langle \mathrm{syz}^r(M_1^{(k)}) \rangle \quad \text{for large } r \\ &= \langle \mathrm{syz}^r(M_1) \rangle \quad \text{for large } r. \end{aligned}$$

So $M_1 \sim_1 M_2^{(k)}$ at the end by (i). Hence $M_2 \sim_1 M_1$ and we are done.

1.10. Corollary. *Suppose R Gorenstein,* $\dim R = d = \dim N$, *and N is C–M. Let*

$$0 \to N \xrightarrow{f} R^n \to T_1 \to 0, \tag{16}$$

$$0 \to N \xrightarrow{g} R^n \to T_2 \to 0, \tag{17}$$

be given, where T_1 and T_2 are such that $S^{-1}T_1$ and $S^{-1}T_2$ are free $S^{-1}R$ modules where $S = R - \bigcup P_i$, $P_i \in \mathrm{Ass}(R)$. Then $T_1 \sim_1 T_2$ on the right.

Proof. Let K be the total quotient ring of R. We apply $\otimes K$, and we get an exact sequence

$$0 \to N \otimes_R K \to K^n \to T_1 \otimes_R K \to 0. \tag{18}$$

Let rank $(N \otimes_R K) = s$. Applying Hom($, K) = {}^*$ we get from (18) an exact sequence

$$0 \to (T_1 \otimes_R K)^* \to (K^n)^* \to (N \otimes_R K)^* \to 0. \tag{19}$$

Let $f_1, \dots, f_n$ be the components of f. Then (19) shows that $f_1, \dots, f_n$ generate $(N \otimes_R K)^*$. Let $f_1, \dots, f_s$ be a free basis of $(N \otimes_R K)^*$. Hence $x \in R - \{0\}$ such that

$$xf_j = \sum_{k=1}^{s} a_{jk} f_k, \quad j = s+1, \dots, n.$$

We apply $\phi : R^n \to R^n$ defined by

$$e_i \mapsto e_i, \qquad i = 1, \dots, s,$$

$$e_j \mapsto xe_j, \quad j = s+1, \dots, n.$$

Then by Lemma 1.3,

$$T_1 = \frac{R^n}{(f_1, \dots, f_n)N}$$

$$\sim_1 \frac{R^n}{(f_1, \dots, f_s, xf_{s+1}, \dots, xf_n)N}$$

$$= \frac{R^n}{(f_1, \dots, f_s, \sum_{k=1}^{s} a_{s+1,k} f_k, \dots, \sum_{k=1}^{s} a_{nk} f_k)N}.$$

Now we apply $\psi : R^n \to R^n$ given by

$$e_1 \mapsto e_1 - a_{s+1,1} e_{s+1},$$
$$e_2 \mapsto e_2 - a_{s+1,2} e_{s+1},$$
$$e_s \mapsto e_s - a_{s+1,s} e_{s+1},$$
$$e_{s+1} \mapsto e_{s+1},$$
$$\vdots$$
$$e_n \mapsto e_n.$$

Then by Lemma 1.4, we get

$$\frac{R^n}{(f_1, \dots, f_s, \sum_{k=1}^{s} a_{s+1,k} f_k, \dots, \sum_{k=1}^{s} a_{nk} f_k)N}$$

$$\xrightarrow{\sim} \frac{R^n}{(f_1, \dots, f_2, 0, \sum_{k=1}^{s} a_{s+2,k} f_k, \dots)N}.$$

Repeating the above operation a finite number of times we get

$$T_1 \sim_1 \frac{R^n}{(f_1, \dots, f_s, 0, \dots, 0)N} = T_1', \text{ say}$$

$$\simeq \frac{R^s}{(f_1, \dots, f_s)N} \oplus R^{n-s} = L_1 \oplus R^{n-s}. \tag{20}$$

So we have

$$0 \to N \to R^s \to L_1 \to 0. \tag{21}$$

Similarly from $0 \to N \to R \to T_2' \to 0$ we get

$$T_2 \sim_1 \frac{R^n}{(g_1, \dots, g_s, 0, \dots, 0)N} = T_2', \text{ say}$$

$$\approx \frac{R^s}{(g_1, \dots, g_s)N} \oplus R^{n-s} = L_2 \oplus R^{n-s}$$

and we have

$$0 \to N \to R^s \to L_2 \to 0. \tag{22}$$

We notice L_1, L_2 are C–M modules with $\dim L_i = \dim r - 1$, $i = 1, 2$. Hence by the theorem $L_1 \sim_1 L_2$ on the right and $L_1 \oplus R^{n-s} \sim_1 L_2 \oplus R^{n-s}$ on the right. Hence $T_1 \sim_1 T_2$ on the right.

1.11. Corollary. *Assume R Gorenstein,* $\dim R = n$. *Suppose that*

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & N_1 & \longrightarrow & R^n & \longrightarrow & M_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi} & & & & & & \\
0 & \longrightarrow & N_2 & \xrightarrow{\psi} & R^n & \longrightarrow & M_2 & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \bigoplus_{k \text{ finite}} \dfrac{R}{(\lambda_k)} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

is exact. Then $M_1 \sim_1 M_2$ *at the end. Here* M_1, M_2 *are C–M modules with* $\dim M_1 = \dim M_2 = \dim R - 1$.

Proof. From the above diagram, we construct T, where T is given by

$$0 \to N_1 \xrightarrow{\psi \cdot \phi} R^n \to T \to 0. \tag{23}$$

Hence we have from the diagram

$$0 \to \bigoplus \frac{R}{(\lambda_i)} \to T \to M_2 \to 0. \tag{24}$$

Thus $T \sim_1 M_2$ at the end. We consider (23) with

$$0 \to N_1 \to R^n \to M_1 0. \tag{25}$$

By the theorem $T \sim_1 M_1$ at the end. Hence $M_1 \sim M_2$ at the end.

1.12. Corollary. *On a Gorenstein ring of dimension* 1, *any module of finite length and finite projective dimension is linked to $R/(x)$ for any n.z.d. $x \in R$.*

Proof. Since (i) M is of finite length, $\operatorname{depth} M = 0$; (ii) $\dim R = 1 = \operatorname{depth} R$; (iii) $\operatorname{Proj\,dim} M + \operatorname{depth} M = \operatorname{depth} R$;
we have $\operatorname{Proj\,dim} M = 1$.

Let x be any n.z.d. of R. Then we get

$$0 \to R \to R \to \frac{R}{(x)} \to 0$$

a projective resolution for $R(x)$. Since $\langle \operatorname{syz}^1(R/(x)) \rangle = \langle \operatorname{syz}^1(M) \rangle$, by the theorem, $M \sim_1 R/(x)$ at the end.

1.13. Corollary. *Assume R Gorenstein,* $\dim R = n$. *Suppose we have*

$$0 \to R^n \xrightarrow{\phi} N \to T_1 \to 0, \tag{26}$$

$$0 \to R^n \xrightarrow{\psi} N \to T_2 \to 0, \tag{27}$$

where T_1, T_2 are C–M modules with $\dim T_i = \dim R - 1$, *for $i = 1, 2$. Then $T_1 \sim T_2$ at the end.*

Proof. Since T_i ($i = 1, 2$), are C–M and $\dim T_i = n - 1$, $\operatorname{Ext}^1(T_i, R) = T_i^{\upsilon}$ ($i = 1, 2$) are also C–M, $\dim T_i^{\upsilon} = n - 1$ and $(T_i^{\upsilon})^{\upsilon} \cong T_i$. From (26) and (27) we have, by applying $\operatorname{Hom}(\ , R) = {}^*$,

$$0 \to N^* \to R^n \to T_1^{\upsilon} \to 0,$$

$$0 \to N^* \to R^n \to T_2^{\upsilon} \to 0.$$

Hence by the theorem $T_1^{\upsilon} \sim_1 T_2^{\upsilon}$ at the end. But since any exact sequence $0 \to L_1 \to L_2 \to R/(x) \to 0$ with L_1, L_2 C–M, $\dim L_i = n - 1$, and x an n.z.d., gives rise to an exact sequence (by applying $*$) $0 \to R/(x) \to L_2^{\upsilon} \to L_1^{\upsilon} \to 0$, we have $T_1^{\upsilon} \sim_1 T_2^{\upsilon} \Leftrightarrow T_1 \sim_1 T_2$ at the end. Hence the result follows.

1.14. Corollary. *Let R be a reduced Gorenstein ring of dimension* 1, *such that every ideal in R can be generated by* 2 *elements. Let* $\bar{R}$ *be the integral closure of R in its full ring of quotients be a finitely generated R-module. Then any module M of finite length on R is weakly linked to* $\bigoplus_{i=1}^{k} R/I_i$ *where the* I_i*'s are ideals of R with* ht $I_i>0$ *for all i. In particular on* $R=K[[x^2,x^{2n+1}]]$ *every module of finite length* $M\sim_1 \bigoplus_{i=1}^{t} R/I_i$, *where* $I_i=(x^{2i},x^{2n+1})$, $1\le i\le n$ *or* I_i *is principal.*

Proof. We consider the following exact sequence

$$0\to S\to R^n\to M\to 0. \tag{28}$$

Since S is torsionless on a one-dimensional reduced Gorenstein ring R such that $\bar{R}$ is a finite type module over R, $S\cong\bigoplus_{\text{finite}} I_i$ where the I_i's are ideals of R ([1], 2, 7). Hence we have

$$0\to S\to R^t\to \bigoplus_{i=1}^{t}\frac{R}{I_i}\to 0. \tag{29}$$

From (28) and (29) by the theorem $M\sim_1\bigoplus R/I_i$. In $R=K[[x^2,x^{2n+1}]]$ since $m=(x^2,x^{2n+1})$ is generated by two elements and R is a domain, R is Gorenstein ([2], 2, 6.4). Since the multiplicity of R with respect to m is 2, every ideal can be generated by 2 elements ([6], 12.8).

Now we use the following lemma. For a proof one is referred to [2], 1, Lemma 1.8.

Lemma. *Let R be a noetherian local integral domain with maximal ideal m and integral closure* $\bar{R}$, *and assume every non-zero ideal of R can be generated by* 2 *elements. Then*

(i) $R_1=m^{-1}$ *is a proper finite integral over the ring of R.*

(ii) *Every non-principal ideal I is an* R_1*-module, i.e.* $R_1 I=I$.

(iii) *If S is a proper finite integral over the ring of R then* $R_1\subseteq S$ *and every ideal of S is generated by at most* 2 *elements.*

Since in our case $\bar{R}=K[[x]]$ is a finite module over R, we have the following unique chain of integral extensions from R to $\bar{R}$:

$$K[[x]]\supset K[[x^2,x^3]]\supset K[[x^2,x^5]]\supset\cdots\supset K[[x^2,x^{2n-1}]]\supset R.$$

By the lemma any finite integral extension of R contained in $\bar{R}$ must be one of those described above, since $(x^2,x^{2k+1})^{-1}=K[[x^2,x^{2k-1}]]$.

Claim. *Any non-principal ideal I of R is isomorphic to* (x^{2i},x^{2n+1}) *for some i.*

Proof. Any non-principal ideal I by the above lemma becomes a module generated by a single element at a certain stage, say at the i-th stage, i.e. over $K[[x^2,x^{2n+1-2i}]]=$

R_i. Since over R_i, I is isomorphic to $I_i=(x^{2i},x^{2n+1})$, which is also principal over R_i, they are isomorphic over R.

Thus by the first part of the corollary the required result follows.

2. Multiplicities

2.1. Proposition. *Let R be a C–M ring of dimension n. Let M_1, M_2 be two C–M modules of dimension r. Then* $M_1 \stackrel{\sim}{\to} M_2 \Leftrightarrow \langle \mathrm{syz}^i(M_1)\rangle \cong \langle \mathrm{syz}^i(M_2)\rangle$ *for some* i, $1\le i<n-r$.

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) We first show that $M\stackrel{\sim}{\to}N \Leftrightarrow \hat{M}\cong\hat{N}$ where $\hat{M}$ is the completion of M with respect to the maximal ideal m of R. $M\simeq N\Rightarrow \hat{M}\simeq\hat{N}$. Let $\hat{M}\stackrel{\phi}{\simeq}\hat{N}$. Then $\phi\in \mathrm{Hom}_{\hat{R}}(\hat{M},\hat{N})=\mathrm{Hom}_R(M,N)\hat{}$. Hence there is a $\phi_0\in\mathrm{Hom}_R(M,N)$ such that $\phi-\phi\in m\,\mathrm{Hom}_R(M,N)\hat{}$. Therefore

$$\frac{R}{m}\otimes_R\frac{N}{\phi_0(M)}=\frac{N}{\phi_0(M)+mN}=0$$

because ϕ and ϕ_0 induce the same map once tensored with R/m. Thus, by Nakayama's Lemma, $N=\phi_0(M)$, i.e. we can map M onto N and similarly we can map N onto M. But for finitely generated modules over commutative rings this implies $M\simeq N$. Thus we are reduced to showing: If R is complete C–M of dimension n, and M_1 and M_2 are two C–M modules of dimension r such that $\langle\mathrm{syz}^i(M_1)\rangle=\langle\mathrm{syz}^i(M_2)\rangle$, then $M_2\simeq M_2$. We consider the following resolution of M_1:

$$0\to S_i\to R^{n_{i-1}}\to\cdots\to R^{n_1}\to R^{n_0}\to M_1\to 0. \tag{30}$$

Let $S_k=\mathrm{syz}^k(M_1)$ given by (30). Since R is complete it has a canonical module Ω. We consider

$$0\to S_1\to R^{n_0}\to M_1\to 0. \tag{31}$$

Since M_1 is C–M with $\dim M_1=r$, and $\dim R=n$,

$$\begin{aligned}\mathrm{Ext}^j(M_1,\Omega)&=0 \quad \text{for } j\neq n-r,\\ &\neq 0 \quad \text{for } j=n-r.\end{aligned}$$

In (31), we apply Hom($,\Omega$), then from the long exact sequence of Ext, we get

$$0\to\mathrm{Ext}^{n-r-1}(S_1,\Omega)\to\mathrm{Ext}^{n-r}(M_1,\Omega)\to 0. \tag{32}$$

Now considering

$$\begin{aligned}&0\to S_2\to R^{n_1}\to S_1\to 0,\\ &\cdots\cdots\\ &0\to S_k\to R^{n_{k-1}}\to S_{k-1}\to 0,\\ &0\to S_i\to R^{n_{i-1}}\to S_{i-1}\to 0,\end{aligned}$$

and applying Hom($,\Omega$), writing the long exact sequence of Ext (as we have done above) we get

$$\mathrm{Ext}^{n-r}(M_1,\Omega)\simeq \mathrm{Ext}^{n-r-1}(S_1,\Omega)\simeq \mathrm{Ext}^{n-r-2}(S_2,\Omega)$$
$$\simeq\cdots\simeq \mathrm{Ext}^{n-r-i}(S_i,\Omega).$$

Similarly $\mathrm{Ext}^{n-r}(M_2,\Omega)\simeq \mathrm{Ext}^{n-r-i}(T_i,\Omega)$, where

$$0\to T_i\to R^{t_{i-1}}\to\cdots\to R^{t_0}\to M_2\to 0$$

is a projective resolution of M_2. Since $\langle S_i\rangle=\langle T_i\rangle$, $\mathrm{Ext}^{n-r-i}(S_i,\Omega)\simeq \mathrm{Ext}^{n-r-i}(T_i,\Omega)$. Hence $\mathrm{Ext}^{n-r}(M_1,\Omega)\simeq \mathrm{Ext}^{n-r}(M_2,\Omega)$. Since for a C–M module M of dimension r, $\mathrm{Ext}^{n-r}(\mathrm{Ext}^{n-r}(M,\Omega),\Omega)\simeq M$, we have from the above $M_1\simeq M_2$.

2.2. The above proposition shows that for any M_1, M_2 of finite length on Gorenstein ring R of dimension n, $M_1\simeq M_2 \Leftrightarrow \langle \mathrm{syz}^i(M_1)\rangle=\langle \mathrm{syz}^i(M_2)\rangle$ for some $i<n$. This naturally gives rise to the following question: What relation exists between M_1, M_2 when $\langle \mathrm{syz}^n(M_1)\rangle=\langle \mathrm{syz}^n(M_2)\rangle$. We have seen by Theorem 1.9 that when $n=1$, $M_1\sim_1 M_2$ at the end. Is this true in higher dimensions also? M. Hochster has the following conjecture which we denote by HC:

HC. On a Gorenstein ring R of dimension n, if M_1, M_2 have finite length and $\langle \mathrm{syz}^n(M_1)\rangle=\langle \mathrm{syz}^n(M_2)\rangle$ then $M_1\sim_n M_2$ at the end.

2.3. Proposition. *Suppose* HC *holds on Gorenstein rings. Then R satisfies the vanishing conjecture.*

Proof. We first prove the following two lemmas.

2.4. Lemma. *Let R be Gorenstein of dimension n. For any two modules M, N with*

$$\mathrm{pd}\, M<\infty,\quad l(M\otimes_R N)<\infty,\quad \dim M+\dim N<\dim R,$$

$\chi(M,N)=0$ *if and only if for any perfect module M and C–M module N such that*

$$l(M\otimes_R N)<\infty,\quad \dim M+\dim N=n-1,$$

$\chi(M,N)=0.$

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) This follows by the the following three claims.

2.5. Claim. *Let M and N be two modules over a C–M ring such that* $l(M\otimes_R N)>\infty$ *and* $\dim M+\dim N<\dim R$. *Then we can choose a system of parameters* $\{x_1,\dots,x_r\}$ *for M contained in* $\mathrm{Ann}_R N$ *such that* $\{x_1,\dots,x_r\}$ *is an R-sequence, where* $r=\dim M$.

Proof. Let $\{P_1, \dots, P_s\} = \mathrm{Ass}(R)$ and $\{q_1, \dots, q_t\}$ be minimal primes of Ass(M). Let $I_N = \mathrm{Ann}_R N$, $I_M = \mathrm{Ann}_R M$. Then since $l(M \otimes_R N) < \infty$ and $\dim M + \dim N \le \dim R$, we can pick

$$x_1 \in I_N - \bigcup_{i=1}^{r} P_i - \bigcup_{j=1}^{t} q_j,$$

noting that since $I_N + I_M$ is m-primary, where m is the maximal ideal of R,

$$I_N \not\subset \left(\bigcup_{i=1}^{r} P_i\right) \cup \left(\bigcup_{j=1}^{t} q_i\right).$$

Then x_1 is an n.z.d. on R and

$$\dim \frac{M}{x_1 M} = \dim M - 1, \qquad \dim \frac{R}{x_1 R} = \dim R - 1.$$

Since $M \otimes_R N \simeq M/x_1 M \otimes_{R/x_1 R} N$ we start with $M/x_1 M$ over $R/x_1 R$ and continue the same process. After a finite number of times we get the required result.

Let $\dim M = r$, $\dim N = s$, $I_M = \mathrm{Ann}_R M$, $I_N = \mathrm{Ann}_R N$. Then $\dim M + \dim R/I_N < n$ and hence $\dim M + \dim R - \mathrm{ht}\, I_N < n$, i.e.

$$\mathrm{ht}\, I_N > \dim M. \tag{33}$$

2.6. Claim. *Suppose we have* $\mathrm{pd}_R M < \infty$, $l(M \otimes N) < \infty$ *and* $\dim M + \dim N < \dim R$. *In order to prove* $\chi(M, N) = 0$ *we can take* N *to be C–M.*

Proof. We choose $\{x_1, \dots, x_r\}$ a system of parameters for M such that $x_i \in I_N$, $i = 1, \dots, r$, and $\{x_1, \dots, x_r\}$ is an R-sequence. We extend it to $\{x_1, \dots, x_r, x_{r+1}, \dots, x_h\}$, a maximal R-sequence contained in I_N where $h = \mathrm{ht}\, I_N = \mathrm{depth}\, I_N$.

We know by [5], Th. 1, that

$$\chi\left(\frac{R}{(x_1, \dots, x_r, \dots, x_k)}, M\right) = 0 \tag{34}$$

for $r < k \le h$. Suppose $\mathrm{depth}\, N = t < n - h$. We map a finite direct sum of $R/(x_1, \dots, x_h)$ onto N; then the kernel N_1 of this map is such that $\mathrm{depth}\, N_1 = \mathrm{depth}\, N + 1$. Repeating this process for a finite number of times we get a module N_{n-h} which is C–M of dimension $n - h$ and such that $\chi(M, N) = 0 \Leftrightarrow \chi(M, N_{n-h}) = 0$ (by (34)). Thus the claim is proved.

Remark. By applying similar arguments we can take M to be perfect.

2.7. Claim. *Under the same hypothesis as in 2.6 we can take* N *to be C–M with* $\dim N = n - r - 1$.

Proof. We have shown in Claim 2.6. that we can take N to be C–M and $\dim N =$

$s<n-r-1$. We consider the following exact sequences:

$$0\to N_1\to\left(\frac{R}{(x_1,\dots,x_{h-1})}\right)^{p_1}\to N\to 0,$$

$$0\to N_2\to\left(\frac{R}{(x_1,\dots,x_{h-2})}\right)^{p_2}\to N_1\to 0,$$

$$\dots\dots$$

$$0\to N_t\to\left(\frac{R}{(x_1,\dots,x_{h-t})}\right)^{p_t}\to N_{t-1}\to 0,$$

where $t=n-r-1-s$. We note each N_i is C–M, $\dim N_i=\dim N_{i-1}+1$, $h-t=r+1$, and $\chi(M,N_{i-1})=-\chi(M,N_i)$. Thus we have constructed a C–M module $N_t=T$ say of dimension $n-r-1$ such that $\chi(M,N)=0 \Leftrightarrow \chi(M,T)=0$. Thus our claim is established; moreover, we have $\dim M+\dim T=r+n-r-1=n-1$.

In the course of proving the three claims we have shown that if M perfect, N C–M, $l(M\otimes N)<\infty$ and $\dim M+\dim N=\dim R-1$ imply $\chi(M,N)=0$, then the vanishing conjecture holds over R.

2.8. Lemma. *Let R be a Gorenstein ring of dimension n. Let M be perfect and N be C–M such that $l(M\otimes_R N)<\infty$, $\dim M+\dim N=n-1$. Then if* HC *holds,* $\chi(M,N)=0$.

Proof. We have seen $\operatorname{ht} I_N=r+1$, where $r=\dim M$. Let $\{x_1,\dots,x_r\}$ be an M-sequence contained in I_N such that it is also an R-sequence. Let

$$0\to R^{n_r}\to\cdots\to R^{n_0}\to M\to 0 \tag{35}$$

be a minimal projective resolution of M. Since $\{x_1,\dots,x_r\}$ is an M-sequence,

$$\operatorname{Tor}_i^R\left(M,\frac{R}{(x_1,\dots,x_r)}\right)=0,$$

therefore applying $\otimes(R/(x_1,\dots,x_r))$ to (35) we get the following exact sequence:

$$0\to\left(\frac{R}{(x_1,\dots,x_r)}\right)^{n_r}\to\cdots\to\left(\frac{R}{(x_1,\dots,x_r)}\right)^{n_0}\to\frac{M}{(x_1,\dots,x_r)M}\to 0. \tag{36}$$

Hence $\operatorname{pd}_{R/(x_1,\dots,x_r)}(M/(x_1,\dots,x_r)M)<\infty$.

Since $(A\otimes_R B)\otimes_R C\simeq A\otimes_R(B\otimes_R C)$ as A,B,C are R-modules, we have

$$\operatorname{Tor}_i^R(M,N)\simeq\operatorname{Tor}_i^{R/(x_1,\dots,x_r)}\left(\frac{M}{(x_1,\dots,x_r)M},N\right).$$

Let

$$S=\frac{R}{(x_1,\dots,x_r)},\qquad Q=\frac{M}{(x_1,\dots,x_r)M}.$$

Then $\chi^R(M, N) = \chi^S(Q, N)$ where Q is a module of finite length and finite projective dimension over S.

Hence we are led to prove the following sublemma.

2.9. Sublemma. *Let R be a Gorenstein ring of dimension n. Let M, N be C–N modules such that $l(M) < \infty$, $\mathrm{pd}_R(M) < \infty$, $\dim N = n-1$. Then $\chi(M, N) = 0$ provided* HC *holds on Gorenstein rings.*

Proof. Suppose HC holds on Gorenstein rings. Since M is a module of finite length and finite projective dimension, $\mathrm{syz}^n(M)$ is free. Again we know (via the Koszul complex) that $\mathrm{syz}^n(R/(x_1, \dots, x_n))$ is also free, where $\{x_1, \dots, x_n\}$ is an R-sequence. Hence by HC, $M \sim_n R/(x_1, \dots, x_n)$. We note $\dim N = n-1$. Now whenever we have

$$0 \to M \to T \to \bigoplus_i \frac{R}{(y_{i1}, \dots, y_{in})} \to 0,$$

the sum being finite, and $\{y_{i1}, \dots, y_{in}\}$ an R-sequence, we get

$$\chi(T, N) = \chi(M, N) + \chi\left(\bigoplus_i \frac{R}{(y_{i1}, \dots, y_{in})}, N\right).$$

But by [5], Th. 1,

$$\chi\left(\bigoplus_i \frac{R}{(y_{i1}, \dots, y_{in})}, N\right) = 0.$$

Hence $\chi(M, N) = \chi(T, N)$. The same argument shows $\chi(M, N) = \pm\chi(M, N)$ for all kinds of linking. Thus when $M \sim_n R/(x_1, \dots, x_n)$ we have

$$\chi(M, N) = \pm\chi\left(\frac{R}{(x_1, \dots, x_n)}, N\right) = 0.$$

2.10. We have shown in 2.2 that to prove the vanishing conjecture on a Gorenstein ring it is enough to prove the following:

Given M a perfect module of finite length, Q a C–M module such that $\dim Q = n-1$, then $\chi(M, Q) = 0$, $n = \dim R$. We choose $\{y_1, \dots, y_{n-1}\}$ a Q-sequence $\mathrm{Ann}_R M$ which is also an R-sequence. We consider an exact sequence $0 \to T \to R^r \to Q \to 0$. Applying $\otimes(R/(y_1, \dots, y_{n-1}))$ we get

$$0 \to \frac{T}{(y_1, \dots, y_{n-1})T} \to \left(\frac{R}{(y_1, \dots, y_{n-1})R}\right)^r \to \frac{Q}{(y_1, \dots, y_{n-1})Q} \to 0, \qquad (37)$$

noting that since $y_1, \dots, y_{n-1}$ is a Q-sequence

$$\mathrm{Tor}_i^R\left(\frac{R}{(y_1, \dots, y_{n-1})R}, Q\right) = 0, \quad i > 0,$$

(by the Special Lemma). Now

$$\chi^R(M,Q)=\chi^{R/(y_1,\dots,y_{n-1})}\left(M,\frac{Q}{(y_1,\dots,y_{n-1})Q}\right).$$

Therefore $\chi^R(M,Q)=0$ if and only if

$$l\left(M\otimes\left(\frac{R}{(y_1,\dots,y_{n-1})R}\right)^r\right)=l\left(M\otimes\frac{T}{(y_1,\dots,y_{n-1})T}\right)$$

(from (37)), i.e. if and only if

$$l\left(M\otimes\frac{T}{(y_1,\dots,y_{n-1})T}\right)=r\left(\frac{T}{(y_1,\dots,y_{n-1})T}\right)l(M)$$

where $r(T/(y_1,\dots,y_{n-1})T)$ is the rank of $T/(y_1,\dots,y_{n-1})T$, which is the rank of

$$\left(\frac{T}{(y_1,\dots,y_{n-1})T}\otimes Q\left\{\frac{R}{(y_1,\dots,y_{n-1})}\right\}\right)$$

where $Q\{R/(y_1,\dots,y_{n-1})\}$ is the total quotient ring of $R/(y_1,\dots,y_{n-1})$.

Thus on a complete intersection R to show $\chi^R(M,Q)=0$ we are led to the following question: On a complete intersection S of dimension 1, given a module M with $l(M)<\infty$, a module T with $T\otimes Q\{S\}$ free over $Q\{S\}$ (the total quotient ring of S) and $\operatorname{Tor}_i^S(M,T)=0$ for all $i>0$, is it true that $l(M\otimes_S T)=r(T)l(m)$ where $r(T)$ is the rank of $T\otimes Q\{S\}$ over $Q\{s\}$?

2.11. Claim. *To prove $\chi^R(M,Q)=0$ it is enough to prove in the above situation $l(M\otimes_S T)\geq r(T)l(M)$.*

Proof. If $l(M\otimes_S T)\geq r(T)l(M)$, from the arguments above we get $\chi^R(M,Q)\leq 0$. Let $\{x_1,\dots,x_n\}$ be a maximal R-sequence contained in $\operatorname{Ann}_R M$. We consider the following exact sequence:

$$0\to L\to\left(\frac{R}{(x_1,\dots,x_n)}\right)^t\to M\to 0. \tag{38}$$

Since $l(M)<\infty$, $\operatorname{pd}(M)<\infty$, $l(L)<\infty$, $\operatorname{pd}(L)<\infty$, from (38) we have

$$\chi(M,Q)+\chi(L,Q)=\chi\left(\left(\frac{R}{(x_1,\dots,x_n)}\right)^t,Q\right)=0 \tag{39}$$

by [5], Lemma 1. Since $\chi(M,Q)\leq 0$ and $\chi(L,Q)\leq 0$, (39) implies we must have $\chi(M,Q)=0$ and $\chi(L,Q)=0$. Hence in this section we investigate the following question: Given a module M of finite length and a module N over a complete intersection of dimension 1, such that $N\otimes Q\{R\}$-free, where $Q\{R\}$ is the total quotient ring of R, is it true that $l(M\otimes_R N)\geq r(N)l(M)$? I do not know the answer in full, but the answer is "yes" in the following cases:

Case i: $M=K=R/m$. Let $r(N)=\operatorname{rank} N=r$, say. Then we have $l(N\otimes_R K)=l(N/mN)=$ minimal number of generators of $N=\mu(N)$ and we know $\mu(N)\geq r(N)$.

Case ii: M is of finite projective dimension. We have

$$0\to N\to R^r\to Q\to 0. \tag{40}$$

Then Q is a module of finite length. Since $\operatorname{depth} M=0$, $\operatorname{depth} R=1$, from $\operatorname{depth} M+\operatorname{pd} M=\operatorname{depth} R$ we get $\operatorname{pd} M=1$. We consider $0\to R^r\to R^r\to M\to 0$ a minimal projective resolution of M. We have

$$0\to \operatorname{Tor}_1^R(M,Q)\to Q^r\to Q^r\to \operatorname{Tor}_0^R(M,Q)\to 0.$$

Hence $\chi(M,Q)=0$. From (40) we have $\chi(M,Q)=rl(M)-l(M\otimes_R N)$. Therefore $l(M\otimes_R N)=rl(M)$.

Case iii: $R=K[[x^2,x^{2n+1}]]$. In this case as we have seen in 1.14, $N\simeq \bigoplus I_i$, the sum being finite, and $I_i=(x^{2i},x^{2n+1})$, $1\leq i\leq n$, and $M\sim_1 \bigoplus R/I_i$ on the right. We first prove the following lemma.

2.12. Lemma. *If* $M\sim_1 M^1$ *on the right and* $l(M^1\otimes_R N)\geq r(N)l(M^1)$, *then* $l(M\otimes_R N)\geq r(N)l(M)$.

Proof. We consider the exact sequences

$$0\to M\to T\to \frac{R}{(x)}\to 0, \tag{41}$$

$$0\to N\to R^r\to Q\to 0. \tag{42}$$

Since Q is of finite length, $\chi(R/(x),Q)=0$. From (41), since $\operatorname{Tor}_2^R(R/(x),Q)=0$,

$$\sum_{i=0}^{1}(-1)^i l(\operatorname{Tor}_i^R(M,Q))=\sum_{i=0}^{1}(-1)^i l(\operatorname{Tor}_i^R(T,Q))=p,$$

say. Now if $l(T\otimes_R N)\geq rl(T)$ that implies (from (42)) $p<0$, and that again implies $l(M\otimes_R N)\geq rl(M)$. Hence when $M\sim_1 M^1$ on the right we have the required result.

So to prove our result it will be enough to take $N=I_i=(x^{2i},x^{2n+1})$ and $M=R/I_k=R/(x^{2k},x^{2n+1})$. We then have to show since $r(N)=1$ that $l(I_i/I_iI_k)\geq l(R/I_k)$ which is true (and easy to check).

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